

MORE AID ABOUT NOTHING

Theorem 1. Any local event that can happen in any arbitrary state of a field can also happen in the vacuum.

Theorem 2 In the vacuum any measurement located at x is maximally correlated with some simultaneous measurement located at y , however far apart x and y may be.

Theorem 3 Every local measurement is infinitely ambiguous, i.e. leaves infinitely many questions unanswered.

NON-RELATIVISTIC QUANTUM FIELD THEORY (NRQFT)

Quantization of the Schrödinger
field: At $t=0$,

$$\psi(\underline{x}) = \frac{1}{(2\pi)^{3/2}} \int d(\underline{k}) e^{i\underline{k} \cdot \underline{x}} d^3 \underline{x}$$

$$\psi^*(\underline{x}) = \frac{1}{(2\pi)^{3/2}} \int d^*(\underline{k}) e^{-i\underline{k} \cdot \underline{x}} d^3 \underline{x}$$

$$[d(\underline{k}), d^*(\underline{k}')] = \delta(\underline{k} - \underline{k}')$$

$$[\psi(\underline{x}), \psi^*(\underline{x}')] = \delta(\underline{x} - \underline{x}')$$

Define $N(\underline{k}) = d^*(\underline{k}) d(\underline{k})$

$$N(\underline{x}) = \psi^*(\underline{x}) \psi(\underline{x})$$

$$N = \int N(\underline{k}) d^3 \underline{k} = \int N(\underline{x}) d^3 \underline{x}$$

$$N_V = \int_V N(\underline{x}) d^3 \underline{x}$$

Then: $[N_V, N_{V'}] = 0$, for disjoint V and V'

N_V has eigenvalues $0, 1, 2, \dots$

$N=0 \Rightarrow N_V=0$ for all subvolumes V
so global vacuum \Rightarrow local vacuum

RELATIVISTIC QUANTUM FIELD THEORY (RQFT)

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Consider charged Klein-Gordon field
At $t=0$,

$$\phi(x) = \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\omega(k)} \left(a(k) e^{i k \cdot x} + b^*(k) e^{-i k \cdot x} \right)$$

$$\omega(k) = \sqrt{m^2 + k^2}$$

$$[a(k), a^*(k')] = \omega(k) \delta(k - k')$$

$$N^+ = \int \frac{d^3k}{\omega(k)} a^*(k) a(k) \quad \dots \text{particles}$$

$$N^- = \int \frac{d^3k}{\omega(k)} b^*(k) b(k) \quad \dots \text{Antiparticles}$$

But if we write $N_V^+ = \int_V N(x) d^3x$
we find $[N_V^+, N_{V'}^+] \neq 0$ for disjoint V and V' .

Similarly for N_V^- .

So N_V^+ does not commute with $N = N^+ + N^-$
and the global vacuum $N=0$

$$\Rightarrow N_V^+ = 0.$$

Two ways out

- ① N_V^+ is not a local observable
we need quantities like Q_V , the
charge in volume V , for which
 $[Q_V, Q_{V'}] = 0$ for disjoint V and V'

But again $N=0 \not\Rightarrow Q_V=0$
(since $[Q_V, N] \neq 0$)

→ local charge
fluctuations.

- ② Write $N_V^+ = \int \underbrace{N_{nw}^+(\xi)}_{\text{Wentzel-Wigner}} d\xi$
where $N_{nw}^+(\xi) = \psi_{nw}^*(\xi) \psi_{nw}(\xi)$
and $\psi_{nw}(\xi) = \frac{1}{(2\pi)^{3/2}} \left(\frac{d^3k}{\omega(k)} \right) a(k) e^{i(k \cdot \xi)} \sqrt{\omega(k)}$

Then

$$[\psi_{nw}(\xi), \psi_{nw}^*(\xi')] = \delta(\xi - \xi')$$

and hence

$$[N_{v, nw}^+, N_{v', nw}^+] = 0$$

for disjoint v and v' .

But $\psi_{nw}^*(\xi)$ cannot be

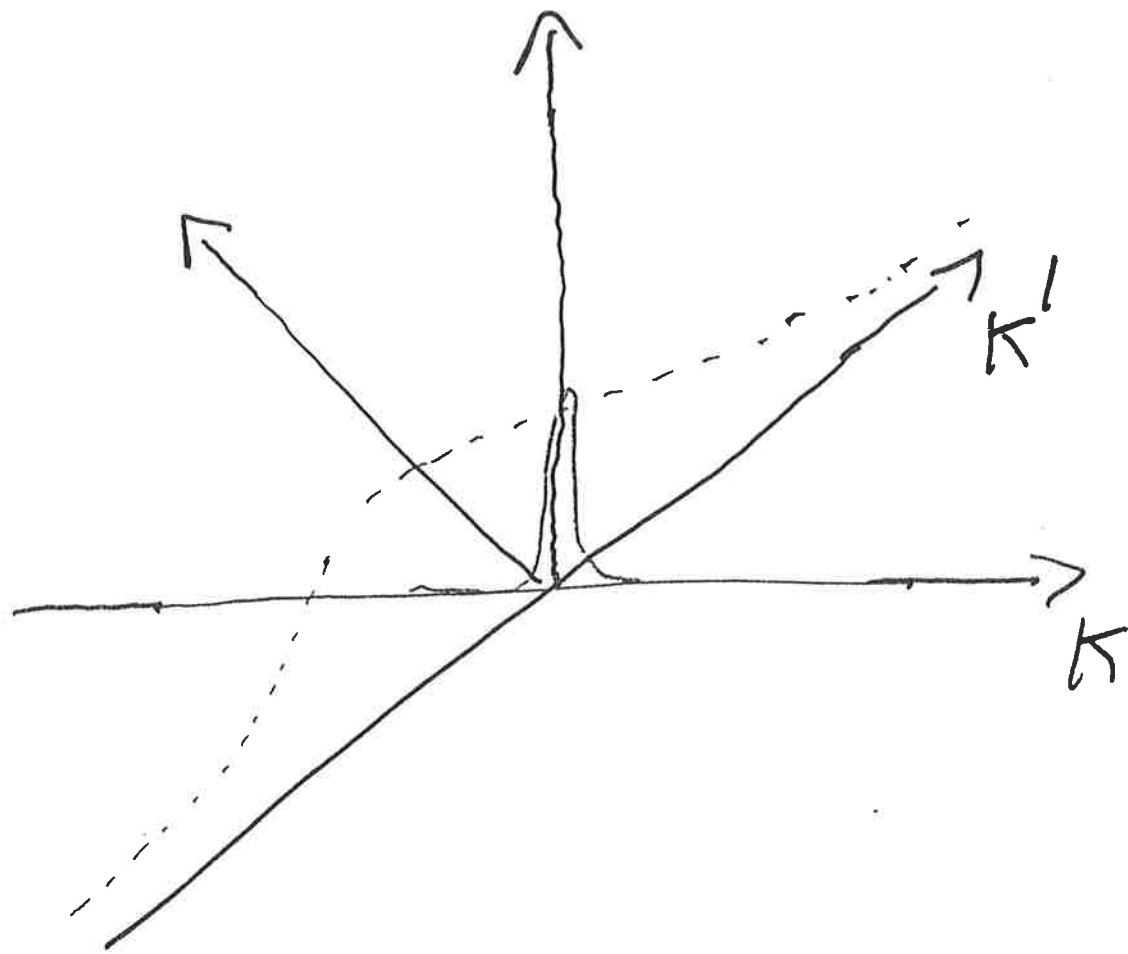
interpreted as creating a particle localized at the point ξ , because we also have-

$$\langle \psi_{nw}^*(\xi) \Omega | \Omega(\underline{x}) | \psi_{nw}^*(\xi) \Omega \rangle \neq 0$$

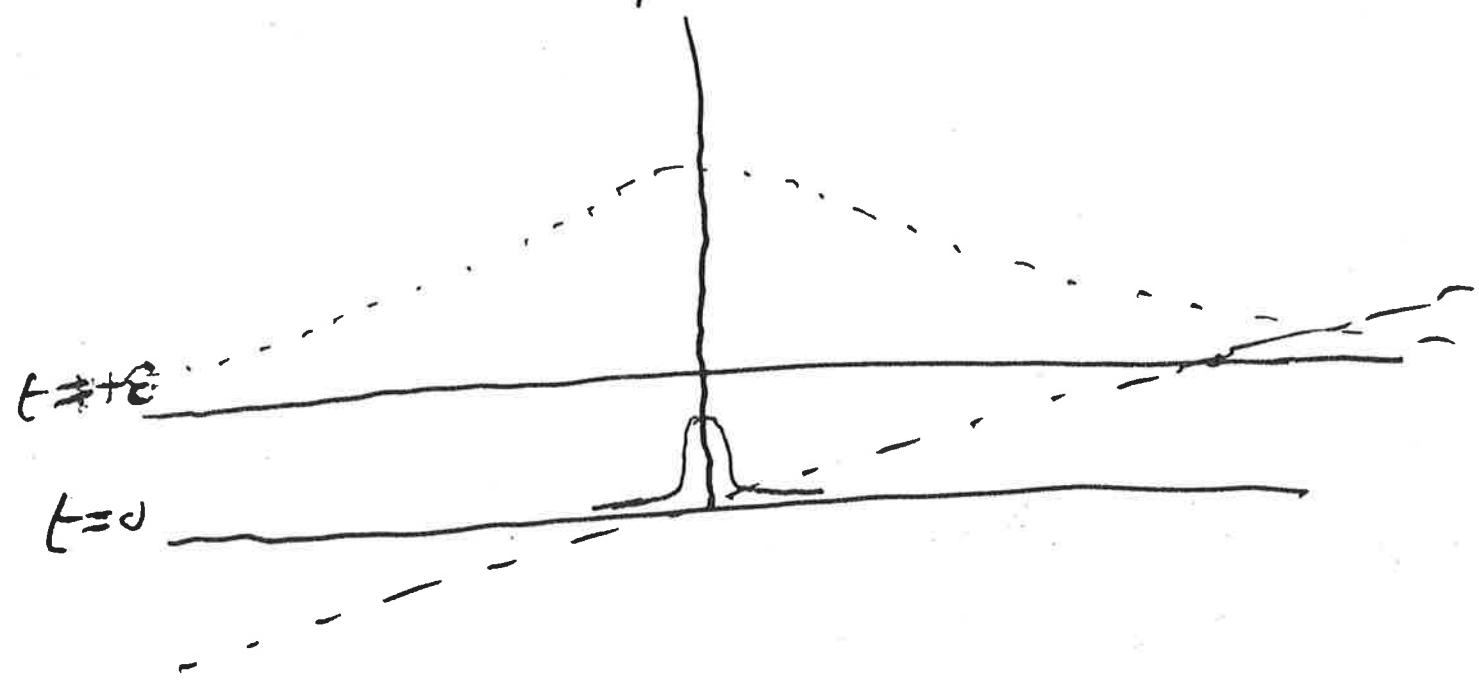
for $\xi \neq \underline{x}$.

(Ω is the vacuum state)
nw localization is 'spread out' in physical \underline{x} -space.

This arises because nw states are non localized on inclined hypersurfaces



This diagram explains
Hegerfeldt's Paradox (1974)



ALGEBRAIC QUANTUM

FIELD THEORY

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$$O \mapsto R(O)$$

\hookrightarrow bounded
open set
in space time

Von Neumann
algebra of
observables

R acts on Hilbert space \mathcal{H}

$$R(O + \underline{a}) = U(\underline{a}) R(O) U^*(\underline{a})$$

\hookrightarrow representation of
translation $x \rightarrow x + \underline{a}$

For time like translations $U(\underline{a})$ is
exponentiated to obtain a Hamiltonian
operator which is non-negative.

Isotony For any two bounded open
sets O, O' , $O_1 \subseteq O_2 \Rightarrow R(O_1) \subseteq R(O_2)$

Locality If O_1 and O_2 are spacelike
related, then $\forall A_1 \in R(O_1), \forall A_2 \in R(O_2)$
 $[A_1, A_2] = 0$

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The global algebra R is the smallest von N. algebra containing all the local algebras. We assume that ρ ^{the representation of} R is irreducible and generated by the translates of $R(0)$ for any 0 .

The Vacuum Ω is the unique state which is invariant under all translations

The Reeh-Schlieder Theorem

Ω is cyclic with respect to \mathcal{H} for any $R(0)$

This just means $\{A\Omega : A \in R(0)\}$ is dense in \mathcal{H} .

Corollary

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Ω is a separating vector for $R(0)$

This just means

$$\forall A \in R(0), A\Omega = 0 \Rightarrow A = 0.$$

I am now going to prove the R-S theorem and its corollary for a very simple analogue of a field theory, in which spacetime collapses to two points and the Von N. algebras are just the algebras of operators on a 2-dimensional Hilbert space. This is just the familiar 2 spin $\frac{1}{2}$ particle system, and for the analogue of the vacuum we shall take initially

$$\Psi_{\text{singlet}} = \frac{1}{\sqrt{2}} \left(|\sigma_{1z}=+1\rangle \otimes |\sigma_{2z}=-1\rangle - |\sigma_{1z}=-1\rangle \otimes |\sigma_{2z}=+1\rangle \right)$$

Then $\forall \phi \in \mathcal{H}_1 \otimes \mathcal{H}_2$, $\exists A_1 \in \mathcal{R}_1$ s.t.

$$|\phi\rangle = A_1 |\Psi_{\text{singlet}}\rangle$$

Proof: By inspection.

$$\begin{aligned} g|\phi\rangle = & \alpha |\sigma_{1z}=+1\rangle \otimes |\sigma_{2z}=-1\rangle \\ & + \beta |\sigma_{1z}=-1\rangle \otimes |\sigma_{2z}=-1\rangle \\ & + \gamma |\sigma_{1z}=-1\rangle \otimes |\sigma_{2z}=+1\rangle \\ & + \delta |\sigma_{1z}=+1\rangle \otimes |\sigma_{2z}=+1\rangle \end{aligned}$$

$$\text{Then } A_1 = \alpha P_1^+ + \beta Q_1 P_1^+ + \gamma P_1^- + \delta Q_1 P_1^-$$

where P_1^\pm projects the state $|\sigma_{1z}=\pm 1\rangle$
and Q_1 rotates spin 1 thro' 180° .

Similarly $\forall \phi \in \mathcal{H}_1 \otimes \mathcal{H}_2$, $\exists A_2 \in \mathcal{R}_2$ s.t.

(11)

Corollary

$$A_1 | \Psi_{\text{singlet}} \rangle = 0$$

$$\Rightarrow A_1 = 0$$

Proof:

By the baby R-S theorem

$\forall \phi \in \mathcal{H}_1 \otimes \mathcal{H}_2$, we can write

$$|\phi\rangle = A_2 | \Psi_{\text{singlet}} \rangle, \text{ so}$$

$$\begin{aligned} A_1 |\phi\rangle &= A_1 A_2 | \Psi_{\text{singlet}} \rangle \\ &= A_2 A_1 | \Psi_{\text{singlet}} \rangle \\ &= 0 \end{aligned}$$

Since ϕ is any vector in $\mathcal{H}_1 \otimes \mathcal{H}_2$, it follows that $A_1 = 0$, (Q.E.D.).

So $| \Psi_{\text{singlet}} \rangle$ is a cyclic vector and a separating vector for R_1 (and similarly for R_2).

we now prove a baby version
of Theorem 1

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Define $p = \text{Prob}^{\Psi_{\text{singlet}}}$ ($P_i \in R_i = 1$)
then $p = \left\| P_i | \Psi_{\text{singlet}} \right\|^2$ — projector.

So $p = 0 \Rightarrow P_i | \Psi_{\text{singlet}} \rangle = 0$
 $\Rightarrow P_i = 0$ (by R-S)

$\therefore P_i \neq 0 \Rightarrow p \neq 0$. QED.

We now turn to a baby version
of Theorem 2

We want to prove.

$\forall P_2, \exists P_1$ s.t. $\langle P_1 P_2 \rangle_{\Psi_{\text{singlet}}}$

$= \langle P_1 \rangle_{\Psi_{\text{singlet}}}$

(i.e. $\text{Prob}^{\Psi_{\text{singlet}}}(P_2=1/P_1=1) = 1$)

Proof Write $\mathbb{P}_{\text{Singer}} = \mathbb{P}_S$

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Write $|\phi\rangle = \mathbb{P}_2 |\psi\rangle / \|\mathbb{P}_2 |\psi\rangle\|$

Then by construction

$$\langle \mathbb{P}_2 | \phi \rangle = 1 \quad \text{--- (1)}$$

But, by the baby R-S theorem

$$|\phi\rangle = C_1 |\psi\rangle \quad \text{--- (2)}$$

where C_1 is some operator on \mathcal{H}_1 ,
(extended to $\mathcal{H}_1 \otimes \mathcal{H}_2$)

Substituting (2) in (1) gives

$$\langle \psi | Q_1 \mathbb{P}_2 | \psi \rangle = 1 \quad \text{--- (3)}$$

where $Q_1 = C_1^* C_1$ is a positive Hermitian operator on \mathcal{H}_1 .

So we can expand

$$Q_1 = \lambda_1 \mathbb{P}_1 + \lambda_1' \mathbb{P}_1' \quad \text{--- (4)}$$

where λ_1, λ_1' are the positive real eigenvalues of Q_1 , and $\mathbb{P}_1, \mathbb{P}_1'$ are orthogonal projections in \mathcal{H}_1 .

Substituting (4) in (3) yields (14)

$$\frac{w_1 \langle \Gamma_1, \rho_2 \rangle_{\mathcal{H}_S}}{\langle \Gamma_1 \rangle_{\mathcal{H}_S}} + \frac{w_2 \langle \Gamma_1', \rho_2 \rangle_{\mathcal{H}_S}}{\langle \Gamma_1' \rangle_{\mathcal{H}_S}} = 1 \quad \dots (5)$$

where

$$w_1 = \lambda_1 \langle \Gamma_1 \rangle_{\mathcal{H}_S}$$

$$w_2 = \lambda_1' \langle \Gamma_1' \rangle_{\mathcal{H}_S}$$

But we know

$$\begin{aligned} \langle Q \rangle_{\mathcal{H}_S} &= \| |c, 1\rangle_{\mathcal{H}_S} \|^2 \\ &= \| |\phi\rangle \|^2 = 1 \end{aligned}$$

$$\therefore w_1 + w_2 = 1 \quad \dots (6)$$

with $w_1 \geq 0, w_2 \geq 0$.

$$\text{Hence LHS (5)} \leq \max \left(\frac{\langle \Gamma_1, \rho_2 \rangle_{\mathcal{H}_S}}{\langle \Gamma_1 \rangle_{\mathcal{H}_S}}, \frac{\langle \Gamma_1', \rho_2 \rangle_{\mathcal{H}_S}}{\langle \Gamma_1' \rangle_{\mathcal{H}_S}} \right)$$

and (5) can only be satisfied if Γ_1 or Γ_1' (or both) satisfy the condition for Theorem 2 (Q.E.D.).

Now Theorems 1 and 2 are trivially true for Ψ_{singlet} . (15)

Theorem 1 just says, all spin components have non-vanishing probability for results ± 1 on either particle (indeed for Ψ_{singlet} all the probabilities are equal to $1/2$!)

while Theorem 2 says all spin components on one particle are maximally correlated with spin components on the other particle. (indeed there are just the mirror-image correlations of Ψ_{singlet} !)

But the proofs of these well-known results for Ψ_{singlet} only used the R-S theorem; so they can be lifted straight back to QFT, with the vacuum replacing Ψ_{singlet} !

In the QFT case, Theorem 2
can be formulated more accurately
as :

For any two spacelike separated
bounded open regions O_1 and O_2
and $\forall \varepsilon > 0, \forall P_2 \in R(O_2)$

$\exists P_1 \in R(O_1)$ s.t.

$$\langle P_1, P_2 \rangle_{\Omega} \geq (1 - \varepsilon) \langle P_1 \rangle_{\Omega}$$

We can also express the maximality
of the correlations specified in Theorem 2
in terms of correlation coefficients.

For any two projectors P_1 and P_2 belonging
to $R(O_1)$ and $R(O_2)$ respectively, we have

$$C(P_1, P_2) = \frac{\langle P_1, P_2 \rangle - \langle P_1 \rangle \cdot \langle P_2 \rangle}{[\langle P_1 \rangle \cdot (1 - \langle P_1 \rangle) \cdot \langle P_2 \rangle \cdot (1 - \langle P_2 \rangle)]^{1/2}}$$

So, for fixed $\langle P_1 \rangle, \langle P_2 \rangle$ 17
 the maximum value of $C(P_1, P_2)$
 is given by

$$C^{\text{max}}(P_1, P_2) = \left[\frac{\langle P_1 \rangle \cdot (1 - \langle P_2 \rangle)}{\langle P_2 \rangle \cdot (1 - \langle P_1 \rangle)} \right]^{1/2} \quad \text{--- (1)}$$

This only attains the value 1
 when $\langle P_1 \rangle = \langle P_2 \rangle$

This condition is satisfied for
 $[2]_{\text{singlet}}$, but Theorem 2 in
 no way depends on this condition.
 We now want to compare (1)
 with the well-known Fredenhagen
bound on correlation coefficients
 (Fredenhagen 1985).

This reads

$$C(P_1, P_2) \leq e^{-m \ell} \sqrt{(1 - \langle P_1 \rangle_{\mathcal{R}}) \cdot (1 - \langle P_2 \rangle_{\mathcal{R}})}$$

where m is mass-gap, and ℓ the minimum

Comparing (1) and (2),
consistency requires

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$$\langle P_1 \rangle_R \leq \frac{e^{-2ml} \langle P_2 \rangle_R}{(1 - \langle P_2 \rangle_R)^2} \quad \dots (3)$$

i.e. for a fixed value of $\langle P_2 \rangle_R$,
the maximally correlated P_1
must have a probability of
occurring that falls off
exponentially with the distance
between O_1 and O_2 .

This result shows how difficult
it would be to observe the
long-range correlations in the
vacuum. But, of course, it
does not show that they don't
exist!

Turning to Theorem 3, the ambiguity referred to arises in the $\mathcal{H}_{\text{singlet}}$ case from the fact that the local projectors are all two-dimensional (i.e. of the form $P_i \otimes I_2$ etc)

In QFT the technical formulation of Theorem 3 is:

$\forall P \in \mathcal{R}(O), P$ is infinite-dimensional

Proof By Driessler's Theorem (1975) the Von N. algebra associated with an unbounded wedge in spacetime is a type III factor.

But every bounded open region is contained in some wedge.

So, by isotony, $\mathcal{R}(O)$ is always a sub-algebra of a type III factor. But in a type III factor all the projectors are infinite-dimensional.

Hence all the projectors in $R(\mathcal{O})$ are infinite-dimensional
Q.E.D.

N.B. This result does not demonstrate that every local algebra is type III — this still remains an open question.

As a corollary of Theorem 3 we can state:

It is never a local question to ask

"Are we in the vacuum state or indeed in an N -particle state (i.e. orthogonal to the vacuum)?"

This raises the fundamental question:

What do (local) particle detectors detect?

The answer is they cannot strictly speaking be detecting particles. They detect certain types of field excitation, which for all practical purposes may resemble particles. \rightarrow FAPP

But in reality (if you will excuse the phrase!) QFT is not a theory of particles, but a theory of fields and their local excitations, and that is all there is to it.
